Point selection strategies for Lepp-based refinement algorithms and new (non Delaunay) Lepp-based algorithms

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Abstract—We present an empirical study of point insertion strategies to obtain good quality 2-dimensional and 3-dimensional meshes. The quality of the elements of the final meshes are measured by using quality metrics based on the shape of the elements and the minimal interior angle.

Keywords—centroid; Lepp; Lepp-bisection; Lepp-centroid; longest edge; meta-centroid; midpoint; refinement; terminal star; terminal tetrahedron, terminal triangle.

I. INTRODUCTION

Triangular mesh generation has been extensively studied and used by engineering and numerical analysts since the seventies for finite element applications [6]. Finite element methods are numerical techniques for the practical analysis of complex physical problems modeled by partial differential equations [12].

Point selection strategies and point insertion methods are key issues to generate new mesh refinement and improvement algorithms to obtain good quality meshes which are useful to be used in finite element methods. Some mesh refinement algorithms such as Lepp-bisection [6] inserts the midpoint of the longest edge of a bad quality triangle and Lepp-Delaunay algorithm performs the Delaunay insertion of the centroid of a terminal quadrilateral formed by two terminal triangles [8].

Others strategies for mesh generation and mesh improvement has been proposed in the literature. Delaunay-based algorithms to generate good quality triangular meshes were proposed for Chew [2], Ruppert [10] and Shewchuk [12] based on inserting the circumcenter of the bad quality triangles.

To improve tetrahedral meshes, Du [4], [5] proposed the centroidal Voronoi tessellation based on Delaunay triangulations as an optimal distribution of the new points. Shewchuk [11] proposed an algorithm that generates a conforming Delaunay mesh of tetrahedra whose circumradius-to-shortest edge ratios are not greater than a certain bound. Chew [3] proposed a mesh generation technique for producing tetrahedral meshes based on Delaunay triangulations which avoids bad quality elements such as sliver tetrahedra. Rodriguez and Rivara proposed serial and parallel 3D Lepp-bisection algorithms to refine tetrahedral meshes [9] which insert the midpoint of the longest edge of the bad quality tetrahedra.

In this work we present an empirical study of point insertion strategies to obtain good quality 2-dimensional and 3-dimensional meshes. These point insertion strategies are combined with the Lepp and terminal edge concepts [7] to generate new refinement and improvement algorithms. These new algorithms assure to eliminate the bad quality elements (needle, cap, etc) and to improve the interior angles of triangular meshes. To improve tetrahedral meshes we propose the insertion of the meta-centroid into a terminal star [9].

II. LEPP-CENTROID ALGORITHM FOR REFINEMENT AND IMPROVEMENT OF TRIANGULAR MESHES

For any triangle $t_0$ of any conforming triangulation $\tau$, the Lepp of $t_0$, denoted by $Lepp(t_0)$, is the ordered list of all triangles $t_0$, $t_1$, $t_2$, ..., $t_{n-1}$, $t_n$, such as $t_i$ is the neighbor triangle of $t_{i-1}$ by the longest edge of $t_{i-1}$, for $i = 1, 2, ..., n$. A couple of triangles $(t_{n-1}, t_n)$ are terminal triangles if they have a common longest edge $E$ (called terminal edge $E$ of Lepp($t_0$), see Figure 1 (a)). A Lepp of triangle $t_0$ has one terminal edge.

Given a conforming input triangulation $\tau$ and threshold of minimal angle $\theta_{min}$, construct a refined and improved triangulation by inserting the centroid $M$ of the terminal quadrilateral formed by a pair of terminal triangles of the Lepp($t_0$), where $t_0 \in \tau$ is any candidate triangle to be refined (see Figure 1 (b)).

Let $\alpha_0$ be the initial minimal angle of the initial mesh $\tau$, then the final mesh $\tau_f$ is a non-Delaunay mesh and the final minimal interior angle $\alpha_f > \alpha_0$.

A very local edge flipping operation over terminal triangles is performed to improve the internal angles when the terminal triangles do not meet the Delaunay condition [1].

Procedure 1 shows the general algorithm.
Algorithm 1 2DLeppCentroid(τ)
1: Input: τ the input mesh of triangles.
2: Output: Good quality final mesh τf.
3: Find S the set of marked triangles to be refined.
4: while S ≠ ∅ do
5: For each triangle t₀ ∈ S.
6: while t₀ remains in the mesh do
7: Compute Lepp(t₀) and find terminal edge L and the terminal triangles associated (tₙ and/or tₙ₋₁).
8: if L is a boundary edge then
9: Bisect terminal triangle tₙ.
10: else
11: if tₙ and tₙ₋₁ are Delaunay terminal triangles then
12: Compute the centroid M of the quadrilateral terminal formed by the terminal triangles tₙ and tₙ₋₁.
13: Divide the terminal triangles tₙ and tₙ₋₁ by inserting the centroid M.
14: else
15: Flip L.
16: end if
17: end if
18: Update S.
19: end while
20: end while

![Figure 1. (a) Lepp(t₀), and M the centroid of terminal quadrilateral formed by terminal triangles t₄ and t₅; (b) Terminal triangles t₄ and t₅ are refined (divided) to create four new triangles.](image)

A. Behavior of the 2D algorithm

An initial mesh with 2,999,992 vertices and 5,999,945 triangles is used. Table I illustrates the information on the sizes of the initial mesh (M₀) and the final mesh (M_f). The algorithm was executed using a threshold of minimum angle θ_{min} = 30°.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Vertices</th>
<th>Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₀</td>
<td>2,999,992</td>
<td>5,999,945</td>
</tr>
<tr>
<td>M_f</td>
<td>10,929,371</td>
<td>21,841,914</td>
</tr>
</tbody>
</table>

1) Angle distribution: We computed the angle distribution of the initial mesh (M₀) and the final mesh (see Figure 2). Note that the algorithm improves the inner angles and the quality of the elements (see Figure 3). Lepp-Centroid eliminates the angles under the threshold θ_{min} = 30° such as is shown in Figure 2.

![Figure 2. Minimum angle distribution. Initial mesh M₀ versus final mesh M_f.](image)

2) Measuring the quality of the elements: In 2-dimensions, we measure the quality of a triangle t as follows:

\[ q(t) = C \frac{a}{le^2} \]

where \( a \) is the area of the triangle t and \( le \) is the length of its longest edge of t [13], \( C = \frac{4.0}{\sqrt{3.0}} \) and \( q \leq 0 \leq 1 \).

Figure 3 illustrates the distribution of elements according to the ranges of quality \( q \) (0 ≤ q ≤ 1). The bad quality elements (cap, needle, etc) located between ranges 0 to 0.3 were eliminated.

![Figure 3. Quality distribution by range of quality. Initial mesh M₀ versus final mesh M_f.](image)

III. ALGORITHMS FOR REFINEMENT AND IMPROVEMENT OF 3D MESHES

Given a conforming input 3D triangulation τ, construct a locally refined triangulation by inserting the meta-centroid
The meta-centroid is computed as the average of the centroids of the terminal tetrahedra that form a terminal star [9]. The Lepp of a tetrahedron \( t_0 \) (Lepp(\( t_0 \))) has one or more terminal edges and their associated terminal stars. A terminal star is formed by a set of terminal tetrahedra, which share a common terminal edge. The final mesh \( \tau_f \) obtained is a good quality non fully Delaunay mesh where the percentage of bad-shaped elements is lower than the percentage of bad-shaped elements of the initial mesh.

The insertion of the meta-centroid \( mc \) of a terminal star [9] allows to improve the quality of the tetrahedra of the mesh.

In three-dimensions, we measure the quality of a tetrahedron \( t \) as follows:

\[
q(t) = C \frac{v(t)}{\text{le}^3}
\]

where \( v(t) \) is the volume of the tetrahedron \( t \), “\( \text{le} \)” is the length of its longest edge of \( t \) [13], \( C = \frac{12.0}{\sqrt{2.0}} \) and \( q \leq 0 \leq 1 \).

A. Behavior of the 3D algorithm

An initial mesh with 500 vertices and 3,089 tetrahedra is used. Table II illustrates information on the sizes of the initial mesh (\( M_0 \)) and the final mesh obtained from the execution of the 3D Lepp-bisection and 3D Lepp-centroid algorithms, using a threshold of length(\( \text{le} \)), where \( \text{le} \) is the length of longest edge of the tetrahedra.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Vertices</th>
<th>Triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_0 )</td>
<td>500</td>
<td>3,089</td>
</tr>
<tr>
<td>Lepp-bisection</td>
<td>721,345</td>
<td>3,836,907</td>
</tr>
<tr>
<td>Lepp-centroid</td>
<td>590,990</td>
<td>3,136,800</td>
</tr>
</tbody>
</table>

Table II and Figure 4 show that 3D Lepp-centroid algorithm provides a smaller mesh and elements of better quality than 3D Lepp-bisection.

**REFERENCES**


